Problem 1

Classify the following equations according to all the properties we've discussed in Figure 1.1:

(a) $u_t = u_{xx} + 2u_x + u$

(b) $u_t = u_{xx} + e^{-t}$

(c) $u_{xx} + 3u_{xy} + u_{yy} = \sin x$

(d) $u_{tt} = uu_{xxxx} + e^{-t}$

Solution

Part (a)

$$u_t = u_{xx} + 2u_x + u$$
$$u_{xx} - u_t + 2u_x + u = 0$$

This is a linear second-order homogeneous PDE with constant coefficients. Comparing this with the general form of a second-order PDE,

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G,$$

we see that $A = 0$, $B = 0$, $C = 1$, $D = -1$, $E = 2$, $F = 1$, and $G = 0$. Calculate the discriminant: $B^2 - 4AC = 0$. Therefore, $u_t = u_{xx} + 2u_x + u$ is parabolic.

Part (b)

$$u_t = u_{xx} + e^{-t}$$
$$u_{xx} - u_t = -e^{-t}$$

This is a linear second-order inhomogeneous PDE with constant coefficients. Comparing this with the general form of a second-order PDE,

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G,$$

we see that $A = 0$, $B = 0$, $C = 1$, $D = -1$, $E = 0$, $F = 0$, and $G = -e^{-t}$. Calculate the discriminant: $B^2 - 4AC = 0$. Therefore, $u_t = u_{xx} + e^{-t}$ is parabolic.

Part (c)

$$u_{xx} + 3u_{xy} + u_{yy} = \sin x$$

This is a linear second-order inhomogeneous PDE with constant coefficients. Comparing this with the general form of a second-order PDE,

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

we see that $A = 1$, $B = 3$, $C = 1$, $D = 0$, $E = 0$, $F = 0$, and $G = \sin x$. Calculate the discriminant: $B^2 - 4AC = 5 > 0$. Therefore, $u_t = u_{xx} + e^{-t}$ is hyperbolic.

Part (d)

$$u_{tt} = uu_{xxxx} + e^{-t}$$

This is a nonlinear fourth-order inhomogeneous PDE.